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EXISTENCE OF SOLUTIONS OF THE HAMILTON-JACOBI EQUATION IN THE NEIGHBORHOOD OF NONDEGENERATE EQUILIBRIUM POSITION*

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It is shown that, when in the equilibrium position the analytic potential energy is nondegenerate and has a local maximum, then there exist an analytic solution of the Hamilton-Jacobi equation at the zero level of total energy. Existence of a smooth solution in the infinitely differentiable case was earlier proved in /1/ by analyzing motions that are asymptotic to the equilibrium position. Unlike in /1/, the present paper uses the direct method of expanding solutions of the Hamilton-Jacobi equation in power series.

Let us consider the natural mechanical system with n degrees of freedom

$$x^{\cdot}=\frac{\partial H}{\partial y},\quad y^{\cdot}=-\frac{\partial H}{\partial x};\quad x=(x_1,\ldots,x_n),\quad y=(y_1,\ldots,y_n)$$

where x are generalized coordinates and y are the generalized momenta, with the analytic Hamiltonian

$$H=K(x,y)+\Pi(x),\quad K=\frac{1}{2}\sum_{i,j=1}^{n}a^{ij}(x)y_iy_j$$

where $K(x,y)$ is the kinetic and $\Pi(x)$ the potential energy of the system. The isolated critical points of function $\Pi(x)$ represent the isolated equilibrium positions. Without reducing the generality, we assume that $x=0$ is a critical point of function $\Pi(x)$, and that $\Pi(0)=0$. We further assume that $\Pi(x)$ has a local maximum at point $x=0$.

The Hamiltonian has the meaning of total energy, and $H(x,y)=h=\text{const}$ is the first integral of the differential equations of motion. The canonical equations can be replaced by a single Hamilton-Jacobi differential equation in partial derivatives

$$\frac{1}{2}\sum_{i,j=1}^{n}a^{ij}(x)\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_j}+\Pi(x)=h \tag{1}$$

where h is the constant of total energy. Below, we shall consider the existence of an analytic solution of Eq.(1) in the neighborhood of point $x=0$ with $h=0$.

By suitable linear transformation we can obtain that in the new variables (which we shall again denote by x and y) the kinetic and potential energies assume the form

$$K=\frac{1}{2}\sum_{i=1}^{n}y_i^2+\frac{1}{2}\sum_{i,j=1}^{n}A^{ij}(x)y_iy_j,\quad A^{ij}(0)=0$$

$$\Pi=\frac{1}{2}\sum_{i=1}^{n}\lambda_ix_i^2+\overline{\Pi}(x);\quad d^2\overline{\Pi}(x)|_{x=0}=0$$

Taking into account the analyticity of functions $\Pi(x)$ and $A^{ij}(x)$, we write the Hamilton-Jacobi equation in the form

$$\frac{1}{2}\sum_{i=1}^{n}\left(\frac{\partial W}{\partial x_i}\right)^2+\frac{1}{2}\sum_{i,j=1}^{n}\left(\sum_{k=1}^{\infty}A_k^{ij}(x)\right)\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_j}+\sum_{k=3}^{\infty}\Pi_k(x)=0 \tag{2}$$

$$A_k^{ij}(x)=\sum_{\alpha_1+\ldots+\alpha_n=k}a^{ij}_{\alpha_1\ldots\alpha_n}x_1^{\alpha_1}\ldots x_n^{\alpha_n},$$

$$\Pi_k(x)=\sum_{\alpha_1+\ldots+\alpha_n=k}p_{\alpha_1\ldots\alpha_n}x_1^{\alpha_1}\ldots x_n^{\alpha_n}$$

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We seek a solution of the form

$$W = W_2 + W_3 + \dots \quad (3)$$

Substituting its expression into Eq.(2), we obtain

$$\frac{1}{2}\sum_{i=1}^{n}\left(\frac{\partial W_2}{\partial x_i}\right)^2 = -\Pi_2 = -\frac{1}{2}\sum_{i=1}^{n}\lambda_i x_i^2$$

from which

$$W_2 = \frac{1}{2}\sum_{i=1}^{n}\Lambda_i x_i^2, \quad \Lambda_i = \pm\sqrt{-\lambda_i}$$

Let us take $\Lambda_i = \sqrt{-\lambda_i}$. The coefficients of the form

$$W_k = \sum_{\alpha_1+\dots+\alpha_n=k} a_{\alpha_1\dots\alpha_n} x_1^{\alpha_1}\dots x_n^{\alpha_n} \quad (k=3, 4, \dots,)$$

are determined from the recurrent relations

$$\sum_{i=1}^{n}\Lambda_i x_i \frac{\partial W_k}{\partial x_i} = -\Pi_k - \frac{1}{2}\sum_{i=1}^{n}\sum_{l=3}^{k-1}\frac{\partial W_l}{\partial x_i}\frac{\partial W_{k+2-l}}{\partial x_i} - \frac{1}{2}\sum_{i,j=1}^{n}\sum_{m=1}^{k-2}A_m^{ij}\sum_{l=2}^{k-m}\frac{\partial W_l}{\partial x_i}\frac{\partial W_{k+2-m-l}}{\partial x_j}$$

that can be written in the form

$$\sum_{\alpha_1+\dots+\alpha_n=k}(\alpha_1\Lambda_1+\dots+\alpha_n\Lambda_n)\, a_{\alpha_1\dots\alpha_n} x_1^{\alpha_1}\dots x_n^{\alpha_n} = \sum_{\alpha_1+\dots+\alpha_n=k} c_{\alpha_1\dots\alpha_n} x_1^{\alpha_1}\dots x_n^{\alpha_n}$$

where the coefficients of the k-th form $c_{\alpha_1,\dots,\alpha_n}$ depend in a definite way on coefficients of forms Π_l $(l = 2, 3, \dots, k)$ and A_l^{ij} $(l = 1, \dots, k-2)$.

The values of $a_{\alpha_1,\dots,\alpha_n}$ are uniquely determined, if the expression $\alpha_1\Lambda_1 + \dots + \alpha_n\Lambda_n$ is never zero for whole nonnegative α_s such that $\alpha_1 + \dots + \alpha_n = 3, 4, \dots$. If $\lambda_i \neq 0$ $(i = 1, \dots, n)$ which is equivalent to the nondegeneracy of the critical point $x = 0$ of potential energy, then $\alpha_1\Lambda_1 + \dots + \alpha_n\Lambda_n \neq 0$ when $\alpha_1 + \dots + \alpha_n = 3, 4, \dots$ and we have the formal solution of the form (3). We shall show that the formally written power series

$$\frac{1}{2}\sum_{i=1}^{n}\Lambda_i x_i^2 + \sum_{\alpha_1+\dots+\alpha_n=3}^{\infty} a_{\alpha_1\dots\alpha_n} x_1^{\alpha_1}\dots x_n^{\alpha_n}$$

is at the coordinate origin an analytic function. The sum of this series that converges absolutely in some neighborhood of the coordinate origin (we denote it by $W(x)$) is the sought solution.

Remark. Let us take the equation

$$\frac{1}{2}\sum_{i=1}^{n}\left(\frac{\partial W}{\partial x_i}\right)^2 + \frac{1}{2}\sum_{i,j=1}^{n}\left(\sum_{k=1}^{\infty}\varepsilon^k A_k^{ij}(x)\right)\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_j} + \sum_{k=2}^{\infty}\varepsilon^{k-2}\Pi_k(x) = 0 \quad (4)$$

If the analytic solution (4) exists in the cube $G = \{|x_i| \leqslant 1, i = 1, \dots, n\}$ for fairly small ε (it will be of the form $W_2 + \varepsilon W_3 + \varepsilon^2 W_4 + \dots$, then $W = W_2 + W_3 + \dots$) is the analytic solution of Eq.(1) in some neighborhood of the coordinate origin $|x_i| \leqslant \varepsilon$, $1 \leqslant i \leqslant n$.

Theorem. If in the equilibrium position the potential energy of the analytic system has a nondegenerate maximum, then in the neighborhood of the equilibrium position there exists an analytic solution of Eq.(1).

Proof. By virtue of the remark it is sufficient to prove the existence of an analytic solution in the cube G of Eq.(4).

Let us consider the linear space of analytic functions $f: R^n \to R$ which in cube G are represented by the absolutely convergent power series

$$f(x) = \sum_{\alpha_1+\ldots+\alpha_n=2}^{\infty} a_{\alpha_1\ldots\alpha_n} x_1^{\alpha_1}\ldots x_n^{\alpha_n}, \quad a_{\alpha_1\ldots\alpha_n} \in R$$

We specify in that space the norms

$$\|f\|_1 = \sum_{\alpha_1+\ldots+\alpha_n=2}^{\infty} (\alpha_1+\ldots+\alpha_n)\,|a_{\alpha_1\ldots\alpha_n}| < \infty,$$

$$\|f\|_2 = \sum_{\alpha_1+\ldots+\alpha_n=2}^{\infty} |a_{\alpha_1\ldots\alpha_n}| < \infty$$

The conditions of norm determination are here satisfied. We denote by A the space with norm $\|\cdot\|_1$ and by B that with norm $\|\cdot\|_2$. It can be readily shown that A and B are Banach spaces. The inequality

$$\|f_1 f_2\|_2 \leqslant \|f_1\|_2 \|f_2\|_2, \quad f_1, f_2 \in B$$

is evident. It shall be used several times below.

We write Eq.(4) as $F(W, \varepsilon) = 0$. Taking the neighborhood V of point $(W_2, 0) \in A \times R$

$$V = \{(W, \varepsilon) : \|W - W_2\|_1 < a, \ |\varepsilon| < b\}; \quad W_2 = \frac{1}{2}\sum_{i=1}^{n} \Lambda_i x_i^2$$

and consider F to be the mapping of neighborhood V into B. Let us show that F satisfies the conditions of the theorem on implicit function in Banach space /2/.

Indeed

1) $$F(W_2, 0) = 0,$$

2) $$\|F(W, \varepsilon) - F(W_2, 0)\|_2 \leqslant na_1 \|W - W_2\|_1 + (n^2 a_2^2 a_3 + a_4)\,|\varepsilon| < \eta$$

when $\|W - W_2\|_1 < \delta$ and $|\varepsilon| < \delta$.

Here

$$\delta = \min\left\{\frac{\eta}{na_1}, \frac{\eta}{n^2 n_2^2 a_3 + a_4}\right\}, \quad a_1 = a + 2\|W_2\|_1, \quad a_2 = a + \|W_2\|_1$$

$$a_3 = \max_{i,j} \sum_{\alpha_1+\ldots+\alpha_n=1}^{\infty} b^{\alpha_1+\ldots+\alpha_n-1}\,|a^{ij}_{\alpha_1\ldots\alpha_n}|, \quad a_4 = 2\left\|\sum_{k=3}^{\infty} b^{k-3}\Pi_k\right\|_2$$

which proves the continuity of mapping F at point $(W_2, 0)$.

3) It can be shown that in the neighborhood V there exists the derivative $F'_W(W, \varepsilon)$ and

$$\|F'_W(W,\varepsilon) - F'_W(W_2, 0)\| = \sup_{h\in A} \frac{\|F'_W(W,\varepsilon)h - F'_W(W_2,0)h\|_2}{\|h\|_1} \leqslant n\|W - W_2\|_1 + n^2 a_2 a_3 |\varepsilon| < \eta$$

when

$$\|W - W_2\|_1 < \delta, \ |\varepsilon| < \delta; \ \delta = \min\left\{\frac{\eta}{2n}, \frac{\eta}{2n^2 a_2 a_3}\right\}$$

Thus the derivative $F_W'(W, \varepsilon)$ is continuous at point $(W_2, 0)$. Since $\det(\partial^2\Pi/\partial x_i \partial x_j)_{x=0} \neq 0$, the equation

$$\sum_{i=1}^{n} \Lambda_i x_i \frac{\partial u}{\partial x_i} = v, \quad v = \sum_{\alpha_1+\ldots+\alpha_n=2}^{\infty} b_{\alpha_1\ldots\alpha_n} x_1^{\alpha_1}\ldots x_n^{\alpha_n} \in B$$

has the single valued solution

$$u = \sum_{\alpha_1+\ldots+\alpha_n=2}^{\infty} \frac{b_{\alpha_1\ldots\alpha_n}}{\alpha_1\Lambda_1 + \ldots + \alpha_n\Lambda_n} x_1^{\alpha_1}\ldots x_n^{\alpha_n}$$

$$\|u\|_1 \leqslant c\|v\|_2, \quad c = \frac{1}{\min_i \Lambda_i}$$

$$\|[F'_W(W_2, 0)]^{-1} v\|_1 \leqslant c\|v\|_2$$

This means that the operator

$$F'_W(W_2, 0) = \sum_{i=1}^{n} \Lambda_i x_i \frac{\partial}{\partial x_i}$$

has a bounded inverse operator.

Thus by virtue of the theorem on implicit function a solution of Eq.(4) $W = W(\varepsilon)$ exists in space A, when ε is fairly small. The theorem is proved.

Let us give and example of existence of nonanalytic solution if the condition of the theorem is not satisfied. Let

$$2\Pi(x_1, x_2) = -(Ax_1^4 + Bx_1^2x_2^2 + Cx_2^4), \quad A, B, C > 0$$

Obviously $x_1 = x_2 = 0$ is a point of local degenerate maximum of function Π. Equation (2) assumes the form

$$\left(\frac{\partial W}{\partial x_1}\right)^2 + \left(\frac{\partial W}{\partial x_2}\right)^2 = Ax_1^4 + Bx_1^2x_2^2 + Cx_2^4 \tag{5}$$

It can be shown that when coefficients A, B and C satisfy the conditions

$$B \neq 0, \quad 4A \pm 2\sqrt{AC} \neq B, \quad 4C \pm 2\sqrt{AC} \neq B \tag{6}$$

and either $A \neq C$ or $A = C \neq B/2$, the analytic solution of Eq.(5) does not exist.

Passing to polar coordinates $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$ we transform Eq.(5) to

$$\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W}{\partial \varphi}\right)^2 = r^4\Phi(\varphi)$$

$$\Phi(\varphi) = A\cos^4\varphi + B\cos^2\varphi\sin^2\varphi + C\sin^4\varphi$$

We seek a solution of that equation in the form $W = r^3 f(\varphi)$. Then function $f(\varphi)$ satisfies the equation

$$f'^2 + 9f^2 = \Phi(\varphi) \tag{7}$$

If

$$4(A + C) \pm 10\sqrt{AC} = 9B \tag{8}$$

then Eq.(7) has the solution

$$f(\varphi) = a_1\cos^2\varphi + a_2\cos\varphi\sin\varphi + a_3\sin^2\varphi$$

Obviously the coefficients A, B and C can be selected so that conditions (6) and (8) are simultaneously satisfied. Then in input coordinates x_1 and x_2

$$W = \frac{1}{3}(\sqrt{A}x_1^2 + \sqrt{C}x_2^2)\sqrt{x_1^2 + x_2^2}$$

This function is, of course, nonanalytic but belongs to class C^2.

Note that the particular solutions $W(x)$ of the Hamilton–Jacobi equations determine the invariant sets $y = \pm\partial W/\partial x$. On assumptions of the theorem they are filled with trajectories that asymptotically approach the equilibrium position as $t \to \pm\infty$ /1/.

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REFERENCES

1. BOLOTIN S.V. and KOZLOV V.V., On asymptotic solutions of equations of dynamics. Vestn. MGU, Ser. Matem. Mekhan., No.4, 1980.
2. KOLMOGOROV A.N. and FOMIN S.V., Elements of the Theory of Functions and Functional Analysis, Moscow, NAUKA, 1981.

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